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GODEL' S INCOMPLETENESS THEOREM AND DIMENSIONAL LOGIC v1.0

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1) Godel's Theorem
2) Linear Logic versus Dimensional Logic
3) Introducing Hierarchies into Typographical Number Theory
4) Dimensional Mathematical Axioms

My understanding of Kurt Godel's Incompleteness Theorem is derived from Douglas Hofstadter's Godel, Escher , Bach: an Eternal Golden Braid. Let me say that there are not many books like GEB, and leave it at that. If any of the following appears to be an unattributed Hofstadter quote, please assume the attribution.

A formal system consists of a set of axioms and a set of rules to manipulate the axioms. An example of an axiom is "Any two points can be joined by a straight line.", and examples of rules are addition, subtraction and equality. A familiar example of a formal system is Plane Geometry - we have all constructed geometric proofs (also called derivations) in high school using axioms and rules. The resulting "proof" or "proposition" is called a statement – in this case, a statement about Plane Geometry.

Proven statements in formal systems are used to help create other, new statements. This is how new truths are discovered in formal systems – statements build on previous statements. It is therefore critical that statements are rigorously proven to be true, because all new statements using a falsely proven statement in their structure will also be false.

A formal system is called "consistent" when every proof generated using the system's rules and axioms is a true statement – that is, $a^2 + b^2$ will always equal c^2 in Plane Geometry. A formal system is consistent if every proof is a true statement. A formal system that produces a contradiction (P and not P) or a falsehood (1=0) would be "inconsistent".

A formal system is called "complete" when every possible true statement that can be made about the system – in this case, every possible true statement that can be made about Plane Geometry – can be generated in that system using only the axioms and rules of the system. If a formal system produced a statement that could not be proven to be true or false, or if a true statement about the system could not be generated by the system, then the system would be "incomplete".

Typographical Number Theory (TNT) – think of it as a strict form of Mathematics - is a formal system just as Plane Geometry is a formal system, even though TNT operates on a much grander scale than Plane Geometry. Around the turn of the 20th century Alfred North Whitehead and Bertrand Russell wrote The Principia Mathematica (PM for short) which defined all the axioms and rules used to create proofs about natural numbers (positive whole integers – 0, 1, 2 ….etc) in TNT. Until Godel came along it was believed that every true statement about numbers could be produced using the proper axioms and rules in the PM. In other words, Typographical Number Theory was considered both "consistent" and "complete" because it was believed that it could produce every possible true statement about natural numbers.

In 1931 Kurt Godel's Incompleteness Theorem(s) proved that PM was either inconsistent or incomplete. Godel ingeniously generated a statement known as the "G Statement" or "G String". This G Statement is consistent in that it is produced using the proper axioms and rules within PM. However, the statement is obviously false because one part contradicts another part. This meant that TNT, strictly following all its internal rules, produced statements which should be true, but which contradict themselves.

Seen one way, this meant that the PM axioms and rules produced a contradiction, making PM inconsistent. Seen another way, the PM axioms and rules produced a statement that could not be proven or disproven, making PM incomplete. Pick your poison, so to speak. Since mathematics continues to work in all cases except the special cases devised by Godel, the consensus is that PM is consistent, but incomplete until it can account for the G Statement.

Godel's proof cannot be explained in a few sentences, so for the actual proof see GEB. For our purposes, Godel's contradictory proof is analogous to the Epimenides Paradox, which roughly translates to "This statement is false." If the statement is true, then the statement is false. If the statement is false, then the statement is true. The meaning "loops" back on itself – it is a self-referential statement – that is, a statement that makes a statement about itself.

Self-reference is the weakness that Godel discovered in TNT. He devised a mathematical statement that talked about itself, producing a statement with two contexts – two levels (the statement itself, and the statement about the statement). Then he added a third level that talked about the previous two levels – a third context. The result is a statement that talks about itself talking about itself. An amazing feat made more amazing by the fact that one of the levels contradicts another level – in simple terms, the statement said it was both true and false at the same time. By essentially re-creating the Epimenides Paradox in TNT Godel produced a statement that was undecidable, cracking TNT's aura of completeness.

Self-reference introduces more than one context into a logical numerical statement. Levels introduce hierarchies, where hierarchies are perceivable contexts (levels) that include or are included by other contexts. In its present form, TNT is designed to produce statements that are either true or false in a context that does not allow a contradictory statement to be true.

It seems clear that TNT must account for hierarchal statements built from multiple contexts – multiple self-references, multiple Points Of View - or it will remain incomplete. To that end, TNT must account for the dimensional nature of ideas, and add dimensional axioms to PM.

Let's begin at the beginning – the Universe is a formal system that includes all statements, logical and mathematical, produced by itself and by all formal systems less capable than itself. By this very inclusion there are no false statements – that is, there are no perceivable statements that do not exist as part of the larger context of all statements.

There is a discussion among mathematicians and logicians as to whether logic and mathematics are equivalent formal systems, or whether mathematics is a subset of logic (as Godel's Theorem seems to imply). To my mind they are equivalent, but this discussion will not and cannot be resolved until it becomes rooted in the underlying physical structure of logic and mathematics.

"This statement is false." is a construction of ideas, and ideas are composed of energy and can be viewed as physical objects in this Universe. The "meaning" we ascribe to "This statement is false." requires us to interpret it from at least two different Points Of View:
1) If the statement is true, it is false;
2) If the statement is false, it is true.

Each of these POVs is a context related to the statement "This statement is false"; therefore these two POV contexts exist in a relationship with each other (if X is related to Z, and Y is related to Z, then X is related to Y.). Contexts are composed of physical ideas, and relationships among contexts are accomplished through shared physical ideas. More accurately, since ideas can be defined as perceivable dimensions, and perceivable dimensions are composed of energy and mass, relationships are achieved through shared physical dimensions.

In terms of concepts introduced in Dimensional Thinking, "This statement is false" is a Dominant Rule context, and the statements "If the statement is true, then the statement is false." and "If the statement is false, then the statement is true." are the physical Sublevels that actually compose the DR.

When viewed as a "linear" statement, where only one interpretation is permitted, we are faced with a paradox. But in fact "This statement is false" requires both sublevels for its physical construction. Viewing statements as physical objects moves us from linear interpretations to dimensional interpretations. The dual sublevel POV requirement gives "This statement is false." its dimensional nature.

The key differentiator between linear interpretations and dimensional interpretations is the physical aspect of ideas. The physical sublevels of a Dominant Rule are the actual composing objects of the physical Dominant Rule – remove or change any dimension of a sublevel and you change the meaning of the DR. In a Universe of dimensions, with each dimension directly or indirectly a node of every other dimension, "meaning" is wholly dependent on the logical collection of dimensions isolated by a Point Of View.

"This statement is false." is a meta-statement because more than one POV is required for its interpretation. It cannot be interpreted on a single linear level; instead, at least two linear levels must be integrated into a meta-level that includes both meanings. That meta-level, that meta-statement is a logical construction that can only be understood as a dimensional object.

Shared dimensions are the internal architecture of logic – they are the physical pathways connecting contexts into aggregates of meaning. The physical nature of Logic must be the foundation of any attempt to make TNT both consistent and complete. The path to consistency and completeness lies in either adding new dimensional axioms to TNT, or re-defining existing axioms to include a dimensional interpretation.

So what might a dimensional axiom look like? Nothing exotic, actually. Dimensions are physical objects obeying the same logical and mathematical laws as other physical objects in the Universe. As physical objects, dimensions have a geometrical form and an internal architecture consisting of other dimensions. This cluster of logically and mathematically associated dimensions creates a Geometric Architecture – GA for short.

Dimensional axioms simply assume the term GA when describing numbers. 1 + 1 = 2 becomes "The Geometric Architecture of the number one, plus the Geometric Architecture of the number 1, equals the Geometric Architecture of the number 2." This statement retains the properties of addition, while adding a dimensional nature to natural numbers. We can "see" two architectures joining to create the architecture for the number 2.

Dimensional axioms make TNT both consistent and complete by adding a dimensional interpretation to Godel's Incompleteness Theorem. In effect, dimensional axioms add hierarchies to TNT. We move from linear interpretations to dimensional interpretations. Godel's Theorem is "true" because it was generated using the proper axioms and rules of TNT, but it is "false" because it contradicts itself. However, using a dimensional interpretation it can be both true and false at the same time – a dimensional interpretation that reflects the physical structure of the Theorem – a direct reflection of its Geometric Architecture.

"True" and "False" therefore take on different meanings in a dimensional TNT. A proof is true if it is constructed using proper axioms and rule applications – that is, all logical and mathematical laws are strictly obeyed in constructing the proof. Truth becomes a function of physical structure, not a single logical Point Of View limited by not allowing contradictions.

A false proof is one that does not strictly apply logical and mathematical laws in the construction of the Geometric Architecture. Falsity in this case means the GA of the proof – the organization of the dimensions composing the GA – is not consistent with the Dominant Rule of the GA. Put another way, falsity requires there be sublevels in the architecture that do not share dimensions with the Dominant Rule of the context.

Dimensional axioms in TNT will help Mathematics achieve its potential as a formal system equal in power to the science of Logic. Along the way, dimensional axioms will also expand the power of human thinking.

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